One also has a particular case of the Thm above, when one has ~~[illegible]~~ Cartesian fibrations in spaces, i.e. $p: \mathcal{D} \to \mathcal{C}$ s.t. $\forall x \in \mathcal{C}$ $p^{-1}(x)$ is an ∞-groupoid. (these are sometimes called right fibrations).

In the model of quasi-categories $p: \mathcal{D} \to \mathcal{C}$ is a right fibration if $\forall\ n \geq 1$ the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{D} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{C} \end{tikzcd}$$

has a lift. for all i in $0 < i \leq n$.

Thm 2: For any ∞-category $\mathcal{C}$ one has an equivalence:

$$St : RFib(\mathcal{C}) \underset{\simeq}{\rightleftarrows} Fun(\mathcal{C}^{op}, Spc) : Un$$

where $RFib(\mathcal{C}) \subseteq Cart(\mathcal{C})$ is the subcategory generated by the Cartesian fibrations in spaces.

Just to fix notation (we leave as an exercise to spell out the details of their definitions) one has dual results:

∞-cat. of coCartesian fibrations / $\mathcal{C}$.

$$\begin{tikzcd} =: St: coCart(\mathcal{C}) \arrow[r, shift left, "\simeq"] & Fun(\mathcal{C}, Cat_\infty) : Un \arrow[l, shift left] \\ =: LFib(\mathcal{C}) \arrow[u, hook] \arrow[r, shift left, "St"', "\simeq"] & Fun(\mathcal{C}, Spc) \arrow[u, hook] \arrow[l, shift left, "Un"] \end{tikzcd}$$

∞-cat. of left fibrations / $\mathcal{C}$.

the natural diagrams in this and the dual case commute.

See Mazel-Gee - All about the Grothendieck Construction + A user's guide to co/Cartesian fibrations for nice discussions of this.

<u>Examples:</u> (i) for any object $X \in \mathcal{C}$ one has.

$\mathcal{C}^{/X} \to \mathcal{C}$ is a right fibration and

$St(\mathcal{C}^{/X} \to \mathcal{C}) : \mathcal{C}^{op} \to Spc$ corresponds to a functor $\mathrm{Hom}_{\mathcal{C}}(-, X)$ which is contravariant in $-$.

Similarly, $\mathcal{C}^{X/-} \to \mathcal{C}$ is a left fibration w/
$St(\mathcal{C}^{X/-} \to \mathcal{C}) = \mathrm{Hom}_{\mathcal{C}}(X, -)$.

(ii) More generally, let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-cats.

$\mathrm{Fun}([1], \mathcal{D}) \times_{\mathrm{Fun}(\{1\}, \mathcal{D})} \mathcal{C} \xrightarrow{p} \mathrm{Fun}(\{0\}, \mathcal{D}) \simeq \mathcal{D}$. is a Cartesian fibration.

and a morphism in $\uparrow$ is p-Cartesian iff its image in $\mathcal{C}$ is an isomorphism.

(iii) <u>Q:</u> How to encode $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to Spc$.
as a functor?
<u>Ans:</u> $\exists$ an ∞-category $Tw(\mathcal{C}) \xrightarrow{p} \mathcal{C} \times \mathcal{C}^{op}$, s.t.
p is ~~left fibration~~ a right fibration. whose straightening corresponds to

$Y_0 : \mathcal{C}^{op} \times \mathcal{C} \to Spc$.
$(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

<u>Construction 1:</u> Recall $I \star J$ is the linearly ordered set where $I, J \subseteq I \star J$ are subsets w/ their initial order & $\forall\, i \in I,\ j \in J$ $i \leq j$.
let $Q : \Delta^{\bullet} \to \Delta$
$[n] \mapsto [n] \star [n]^{op} \simeq [2n+1]$.

$$\mathrm{Tw}(\mathcal{C})_n := \mathcal{C}(Q([n])).$$

Eg: $\mathrm{Tw}(\mathcal{C})_0 = \{f : X_0 \to Y_0\}.$

$$\mathrm{Tw}(\mathcal{C})_1 = \left\{ \begin{tikzcd} X_0 \arrow[r] \arrow[d] & Y_0 \\ X_1 \arrow[r] & Y_1 \arrow[u] \end{tikzcd} \right\}$$

$$\mathrm{Tw}(\mathcal{C})_2 = \left\{ \begin{tikzcd} X_0 \arrow[r] \arrow[d] & Y_0 \\ X_1 \arrow[r] \arrow[d] & Y_1 \arrow[u] \\ X_2 \arrow[r] & Y_2 \arrow[u] \end{tikzcd} \right\}.$$

One has a functor $p : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$. (this is a right fibration)

$$X_0 \to Y_0 \mapsto (X_0, Y_0)$$

$$Y_0 := St^R(p) : \mathcal{C}^{op} \times \mathcal{C} \to Spc.$$

Construction 2: Consider the morphism in $\mathrm{Cart}(\mathcal{C})$ given by the diagram:

$$\begin{tikzcd} \mathrm{Fun}([1], \mathcal{C}) \arrow[r, "\alpha := ev_0 \times ev_1"] \arrow[d, "ev_0"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "p_1"] \\ \mathcal{C} \arrow[r, "id_{\mathcal{C}}"'] & \mathcal{C} \end{tikzcd}$$

Applying unstraightening we get:

$$\mathrm{Un}(\alpha) : \text{underslice}_{\mathcal{C}} \overset{\simeq}{\Rightarrow} \mathrm{Const}_{\mathcal{C}}, \quad \text{where}$$

$\mathrm{Un}(ev_0) =: \text{underslice}_{\mathcal{C}} : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$, $X \mapsto \mathcal{C}^{X/}$, $\quad \mathrm{Un}(p_1) =: \mathrm{Const}_{\mathcal{C}} : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$, $X \mapsto \mathcal{C}$.

Notice $Un(\alpha) \in Fun([1], Fun(\mathcal{C}^{op}, Cat_\infty)) \simeq Fun(\mathcal{C}^{op} \times [1], Cat_\infty)$.
$\widetilde{Un}(\alpha) \in Fun(\mathcal{C}^{op}, Fun([1], Cat_\infty)) \simeq$

Moreover, since $ev_1(\widetilde{Un}(\alpha)) = \mathcal{C}$, one has $\widetilde{Un}(\alpha) : \mathcal{C}^{op} \to Cat_{\infty/\mathcal{C}}$.
$\cap$
$Fun([1], Cat_\infty)$.

<u>Claim:</u> $\overline{Un}(\alpha)$ factors as:

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "\widetilde{Un}(\alpha)"] \arrow[dr, "\overline{Un}(\alpha)"'] & LFib(\mathcal{C}) \arrow[d, hook] \\ & Cat_{\infty/\mathcal{C}} \end{tikzcd}.$$

$\overline{Y}_0 := St \circ \overline{U}_n(\alpha) : \mathcal{C}^{op} \to Fun(\mathcal{C}, Spc)$.

$Y_0 : \mathcal{C}^{op} \times \mathcal{C} \to Spc$.

Exercise: Both constructions agree. [HA Prop. 5.2.1.11].

(iv) <u>Adjoint functors.</u>

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between two ∞-categories.
This determines a coCartesian fibration $\widetilde{F} \to [1]$ by unstr.

<u>Def'n:</u> F admits a <u>right adjoint</u> F^R if
$\widetilde{F} \to [1]$ is also a <u>Cartesian</u> fibration, i.e.

$F^R = St^{Cart}(\widetilde{F}) : [1]^{op} \to Cat_\infty$. , which ~~can~~ correspond to.

$F^R : \mathcal{D} \to \mathcal{C}$.

One can similarly make sense of left adjoints.

- Equivalence of ∞-categories

The following could be called the fundamental theorem of ∞-categories.

Thm: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-cats, then F is an equivalence* if and only if the following hold:

- F is fully faithful, i.e. $\forall\, X, Y \in \mathcal{C}$ the map
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$ is an isomorphism
- F is essentially surjective, i.e. $\forall\, X \in \mathcal{D}$ there exists $Y \in \mathcal{C}$ s.t. $F(Y) \simeq X$ in $\mathcal{D}$.

* this means F induces an eq. on $h\mathrm{Cat}_\infty$.

Examples of limits & colimits:

(i) Kan extensions. Consider $\iota: \mathcal{C}_0 \to \mathcal{C}$ a functor & $\mathcal{D}$ another ∞-cat. one has a restriction functor:

$$\begin{tikzcd} (-)\circ\iota : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[r] & \mathrm{Fun}(\mathcal{C}_0, \mathcal{D}). \arrow[l, bend right, "LKE_\iota"'] \arrow[l, bend left, "RKE_\iota"] \end{tikzcd}$$

The left Kan extension LKE_ι is a left adjoint to ι, and the right Kan extension RKE_ι is a right adjoint to ι.

These adjoints are not always defined. The following is a very useful criterion for when these exist:

Lemma: Let $\iota: \mathcal{C}_0 \to \mathcal{C}$ and $F: \mathcal{C}_0 \to \mathcal{D}$ be functors. Then

(a) if $\forall\, X \in \mathcal{C}$ $\operatorname{colim}_{\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{/X}} F$ exists, then $LKE_\iota F$ exists;

(b) if $\forall x \in \mathcal{C}$, $\lim\limits_{\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}^{x/}} F$ exists, then $RKE_z(F)$ exists.

Moreover, in any of these cases we have:

$$LKE_z(F)(x) \simeq \underset{Y \in \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}^{/x}}{\mathrm{colim}} F(Y), \text{ and } RKE_z(F)(x) = \lim_{Y \in \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}^{x/}} F(Y).$$

(ii) The ∞-category Spc has all (small) limits & colimits.

<u>Idea of proof:</u> let $F: K \to Spc$ be a diagram.

Notice. F admits a colimit $\Leftrightarrow$ $LKE_z(F)$ exists where $z: K \to K^{\triangleright} := K \amalg *$ w/ a single morphism from each vertex of K to $*$.

Since $\exists$ $Un(F) \to K$ is a ~~right~~ left fibration classifying F.

The existence of $LKE_z(F)$ is equivalent to the existence of $\widetilde{Un}(F) \to K^{\triangleright}$ a ~~right~~ left fibration s.t. $\widetilde{Un}(F) \times_{K^{\triangleright}} K \simeq Un(F)$.

~~Now the existence of $\widetilde{Un}(F)$ follows~~
So one only needs to argue that $\widetilde{Un}(F)$ exists.
[Cisinski - Prop. 6.1.14]

Similarly, one can argue that all limits exist.

(iii) any presheaf ∞-category, i.e. $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spc})$ has all (small) limits & colimits.

(iv) Cat_∞ has all (small) limits & colimits

To check this claim is a bit trickier. The idea is to use a general comparison between limits & colimits of an ∞-category & homotopy limits & colimits of a simp. model cat. presenting it. Namely:

<u>Prop:</u> Let $F: I \to C$ be a simplicial functor between fibrant simplicial categories.
Consider $N^{h.c.}(F): N^{h.c.}(I) \to N^{h.c.}(C)$ the ~~associated~~ functor between the associated ∞-categories.
Then
- F has a homotopy colimit.

$\Updownarrow$

- $N^{h.c.}(F)$ has a colimit.

A particularly important case is when C is a combinatorial simplicial model category. Then $N^{h.c.}(C^{c.f.})$ is an ∞-category w/ all (small) limits & colimits.

The strategy then is to find a comb. simp. model cat. ~~realizing~~ realizing Cat_∞, see §3.1 [HTT]

(v) Let Spc_* be the category of pointed spaces.

$\Omega: \mathrm{Spc}_* \to \mathrm{Spc}_*$, $X \mapsto * \times_X *$. Then $\mathrm{Sptr} := \lim(\cdots \xrightarrow{\Omega} \mathrm{Spc}_* \xrightarrow{\Omega} \mathrm{Spc}_*)$.

∞-cat. of spectra.